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ABSTRACT

Methods for the calculation of eta coefficient, or correlation ratio, squared have recently been presented for examining the strength of relationship in univariate analysis of variance. This paper extends them to the multivariate case in which the effects of independent variables may be examined in relation to two or more dependent variables, and presents considerations for the calculation, use, and interpretation of the coefficients. The concepts of eta squared are generalized to the multivariate case through the application of the multivariate general linear hypothesis. Both orthogonal and non-orthogonal solutions are considered. (Author)

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Strength of Relationship in Multivariate Analysis of Variance

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The use of the eta (η) coefficient, or correlation ratio, has been suggested for comparing the magnitude of given effects across experiments and assessing the strength of relationship between independent and dependent variables in univariate analysis of variance (UANOVA) situations (Kennedy, in press). Although its latter function has been criticized (Glass & Hakstian, 1969), the procedure is recommended for use because it is independent of sample size, while the F-ratio and its associated probability value is largely a function of sample size (Hayes, 1963). UANOVA refers to experiments in which several independent variables may be employed in the design, but only one dependent measure. This paper exanines the notion of eta squared in the multivariate case (MANOVA) in which the effects of independent variables may be investigated in relation to two or more dependent measures, and presents considerations for the calculation, use, and interpretation of the coefficients.

UANOVA Case

Two methods for the calculation of eta squared (η^2) in UANOVA have recently been presented. In each case, eta squared provides a descriptive index of the proportion of variance in the dependent variable that is accounted for by the independent variable. Kerlinger (1964) has proposed that

$$\eta^2 = SS_{ho}/SS_Y, \tag{1}$$

where SS_{ho} denotes the sum of squares for a particular hypothesis and SS_Y represents the total sum of squares in the dependent variable.

Another computational formula has been proffered by Cohen (1966) and Friedman (1968). When the univariate F-ration (UFR) is greater than one,

$$\eta^2 = df_{ho}UFR/df_{ho}UFR + df_e \tag{2}$$

which reduces to

$$\eta^2 = SS_{ho}/SS_{ho} + SS_e. \tag{3}$$

Since $SS_Y = SS_{ho} + SS_e$ in the one-way UANOVA case, equations (1) and (3) are equivalent. Differences in the two approaches when more than one independent variable is involved will be discussed after the multivariate analogues have been presented.

MANOVA Case

The concept of eta squared in UANOVA can be generalized to the multivariate case through application of the

multivariate general linear hypothesis

$$\underset{c\ v}{Y.} = \underset{c\ pp}{X}\ \underset{p\ v}{\beta} + \underset{c\ v}{E.}, \qquad (4)$$

where Y. is a matrix containing the means on the dependent variables for each cell in the design, X is a design matrix containing "1's" and "0's" corresponding to the presence or absence of a particular effect in the model, and β is a matrix of parameters or effects to be tested. The elements of E. are assumed to be distributed in multivariate normal fashion with mean ${}_{c}0_{v}$, and variance-covariance matrix ${}_{c}D_{c}^{-1} \times \underset{v\ v}{\Sigma}$, where D is a diagonal matrix containing the cell sizes. For a more complete discussion of the general linear hypothesis see Graybill (1961), Hendenhall (1968), and Morrison (1967).

Least square estimates of effects is given by

$$\underset{p\ v}{\hat{\beta}} = (\underset{p\ c}{X'}\underset{c\ c}{D}\underset{c\ v}{X})^{-1}\underset{p\ c}{X'}\underset{c\ c}{D}\underset{c\ v}{Y}. \qquad (5)$$

In experimental design applications of equation (4), the rank (r) of X is typically less than the number of parameters (p). Each parameter, then, is not estimable since (X'DX) in equation (5) can not be inverted. One solution is to reparameterize X to create r linear combinations of the original parameters in β which are of concern to the researcher. This can be accomplished by factoring X into

$$\underset{c\ p}{X} = \underset{c\ r}{K}\ \underset{r\ p}{L}, \qquad (6)$$

where the rank (K)= rank (L)= rank (X)= r. Equation (4)

can then be re-written as

$$\underset{c\ v}{Y.} = \underset{c\ r}{K}\ \underset{p\ v}{L}\ \underset{c\ v}{\beta} + \underset{c\ v}{E.} = \underset{c\ r}{K}\ \underset{r\ v}{\beta^*} + \underset{c\ v}{E.}, \qquad (7)$$

$\beta^* = L\beta$. Least square estimates of the linear combinations of the original parameters can be obtained by

$$\underset{r\ v}{\hat{\beta}^*} = (\underset{r\ c}{K'}\ \underset{c\ r}{D}\ \underset{r\ c}{K})^{-1}\ \underset{r\ c}{K'}\ \underset{c\ v}{D}\ \underset{c\ v}{Y.} \qquad (8)$$

The total sum of squares and cross-products matrix for all r effects is given by

$$\underset{v\ r}{\hat{\beta}^{*\prime}}\ \underset{r\ c}{K'}\ \underset{c\ c}{D}\ \underset{c\ v}{Y.} \qquad (9)$$

The sum of squares and cross-products matrix for each of the r effects is obtained by successively entering each column of $\hat{\beta}^{*\prime}$ and row of K'. The partitioning can be expressed as

$$\begin{bmatrix} \underset{v\ r_1}{\hat{\beta}^{*\prime}}\ \underset{c\ c}{K'D}\ \underset{c\ v}{Y.} \\ \hline \underset{v\ r_j}{\hat{\beta}^{*\prime}}\ \underset{c\ c}{K'D}\ \underset{c\ v}{Y.} \\ \hline \vdots \\ \underset{v\ r_n}{\hat{\beta}^{*\prime}}\ \underset{c\ c}{K'D}\ \underset{c\ v}{Y.} \end{bmatrix} \qquad (10)$$

where r_1, r_j, and r_n refer to the first, j-th., and last column and row of $\hat{\beta}^{*\prime}$ and K' respectively, and correspond to the sum of squares and cross-products for the first, j-th., and last effect in the model.

In designs with one subject per cell, D= I and

does not appear. In situations where the linear combinations of the original variables of interest to the investigator are not orthogonal or the number of subjects in the cells of the design are not equal, an additional reparameterization of equation (7) is necessary in order to obtain appropriate sum of squares and cross-products for the r effects in the model. This is accomplished by factoring K into

$$K = K^{\perp} T' , \qquad (11)$$
$$c\,r \quad c\,r \quad r$$

where $K^{\perp}$ is orthonormal in the metric D, and T' is the Cholesky factor of K. Equation (7) can then be re-written as

$$Y. = K^{\perp} T' \beta^{*} + E. = K^{\perp} \alpha + E., \qquad (12)$$
$$c\,v \quad c\,r \; r\,v \quad c\,v \quad c\,r \; r\,v \quad c\,v$$

where $\alpha = T'\beta^{*}$. Least square estimates of effects can be obtained by

$$\hat{\alpha} = (K^{\perp\prime} D K^{\perp})^{-1} K^{\perp\prime} D Y. = K^{\perp\prime} D Y., \qquad (13)$$
$$r\,v \quad r\,c \quad c\,r \qquad r\,c \quad c\,v \quad r\,c \quad c\,v$$

and the sum of squares and cross-products is given by

$$\hat{\alpha}' K^{\perp\prime} D Y. = \hat{\alpha}' \hat{\alpha}. \qquad (14)$$
$$v\,r \quad c\,c\,v\,v \quad r\,v$$

By entering successive columns and rows of $\hat{\alpha}'$ and $\hat{\alpha}$ respectively, as in (10), the sum of squares and cross-products for each effect can be obtained. Although these effects are not independent, they can be examined in a step-wise fashion according to the orthogonal reparameterization suggested above. The sum of squares and cross-products for a particular r_j effect will be appropriate

when effects r_{j+1}, r_{j+2},.....r_n=0. See Bock (1966), Bock & Haggard (1968), Rulon & Brooks (1968), and Morrison (1967) for a discussion of multivariate tests of effects.

The univariate and multivariate sum of square equivalents are presented in Table 1.

Insert Table 1 about here

Although conceptually identical, they do differ with respect to their applicability. In UANOVA, strength of relationship is directly calculable from the sums of squares obtained in the process of testing a hypothesis. This is not possible in the MANOVA case because the dependent variables in the design are not independent. Here, the calculation of strength of strength of relationship requires an adjustment of the original sums of squares among the dependent variables based on their covariation. For each hypothesis, the procedure should permit the examination of the sum of squares for the second variable eliminating the first, the p-th. eliminating the previous ones, and so on down to the v-th. dependent variable.

To express this adjustment process in general, let the sum of squares and cross-products for error and for the hypothesis plus error in both the orthogonal and non-orthogonal case be represented by S_e and S_{th}, respectively. The rows and columns of S_e and S_{th} can

then be symbolized as follows;

$$\begin{array}{c} \\ 1 \\ 2 \\ \cdot \\ \cdot \\ p \\ \cdot \\ \cdot \\ v \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & \cdots p & \cdots v \end{array} \\ \begin{bmatrix} s_{e_{11}} & s_{e_{12}} & s_{e_{1p}} & s_{e_{1v}} \\ s_{e_{21}} & s_{e_{22}} & & \\ & & & \\ s_{e_{p1}} & & s_{e_{pp}} & \\ & & & \\ s_{e_{v1}} & & & s_{e_{vv}} \end{bmatrix} \end{array} \tag{15}$$

$$\begin{array}{c} \\ 1 \\ 2 \\ \cdot \\ \cdot \\ p \\ \cdot \\ \cdot \\ v \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & \cdots p & \cdots v \end{array} \\ \begin{bmatrix} s_{th_{11}} & s_{th_{12}} & s_{th_{1p}} & s_{th_{1v}} \\ s_{th_{21}} & s_{th_{22}} & & \\ & & & \\ s_{th_{p1}} & & s_{th_{pp}} & \\ & & & \\ s_{th_{v1}} & & & s_{th_{vv}} \end{bmatrix} \end{array} \tag{16}$$

Consider the first two rows and columns of these matrices. The reduced sum of squares for error and "total" for the second variable eliminating the first is, respectively

$$s^*_{e_{22}} = s_{e_{22}} - s_{e_{21}} s^{-1}_{e_{11}} s_{e_{12}}, \text{ and} \tag{17}$$

$$s^*_{th_{22}} = s_{th_{22}} - s_{th_{21}} s^{-1}_{th_{11}} s_{th_{12}}. \tag{18}$$

The adjusted hypothesis sum of squares for the second variable is obtained by

$$s^*_{h_{22}} = s^*_{th_{22}} - s^*_{e_{22}}. \tag{19}$$

Additional dependent measures can be successively examined in the same fashion. The method is similar to that employed in the analysis of covariance, stepwise regression, and the "step-down" analysis suggested by Roy & Bargmann (1958), except that the tests of significance need not be carried out.

The multivariate equivalents of the Kerlinger and Cohen-Friedman formulae can then be represented, respectively, as follows;

$$\eta^2 = 1/v \sum_{p=1}^{v} s^*_{ho_p} / s_{Y_p}, \text{ and} \qquad (20)$$

$$\eta^2 = 1/v \sum_{p=1}^{v} s^*_{ho_p} / s_{ho_p} + s_{e_p}. \qquad (21)$$

When only one dependent measure is employed, equations (20) and (21) naturally reduce to their appropriate UANOVA form.

In one-way MANOVA, each yield the same value, indicating the proportion of variability among the dependent measures that is explained by the independent variable. When more than one factor is included in the design, the Cohen-Friedman value will vary as a function of their number and magnitude since there is likely to be a proportionate reduction in error variability. Thus, comparisions of the effect of similar independent variables among experiments will be difficult unless the same number and kind of factors

are included in the designs being compared (Kennedy, in press; Fleiss, 1969). Here, the use of the Kerlinger formula is suggested.

The notion of eta squared advanced here appears useful in the fixed effects MANOVA case generally, for the purpose previously mentioned, and specifically, for assessing the strength of relationship in a given study when all of the levels of interest in the independent variable are completely represented in the design.

Concerning the dependent measures, the formulae do not yield values that are invariant under alternate orderings. Logically, measures should be arranged in terms of their importance to the study under investigation. If conceptually newer as well as older dependent measures are included in the design, the former should be ordered last, placing the burden of proof on them to demonstrate their meaningfulness in relation to the effect being investigated. To the extent that the dependent measures in the design are correlated, the magnitude of the eta squared value will be lower and the variability among the values obtained under different orderings will be greater. It follows, then, that when the dependent variables have minimum intercorrelation, the interpretability and the value of eta squared will be maximized.

Thus, the use of the coefficient in MANOVA does not mean license to include a potpourri of unscrupulously defined measures, but points to the integration of experimental design with the crucial issue is psychological testing and measurement, namely, construct validity. It also focuses on a factor of experimental invalidity that is rarely mentioned in the literature on design. Certainly any independent variable worth investigating is likely to have effects that go beyond what is normally embodied in a single measure of outcome. Studies employing one measure must be regarded as limited and, therefore, less valid than comparable experiments in which a number of outcomes are examined (Bracht & Glass, 1968). Regarding the latter, eta squared appears particularly useful since it permits the evaluation of an effect in relation to both comprehensiveness and potency. For example, a UANOVA value of 50% indicates that the effect possesses relative potency, but provides no information concerning its comprehensiveness or coverage since only one dependent measure is included. However, an obtained MANOVA value of 50% is equally potent, as well as a general index of the comprehensiveness of the effect. Thus, the use of the coefficient in MANOVA situations supplies important, additional information not provided by its application in UANOVA.

Table 1
Sum of Square
Equivalents

UANOVA Case	MANOVA Case	
	Orthogonal	Non-Orthogonal
SS_{ho}	$\hat{\beta}^{*\prime} K'D Y.$ v r_j c c v	$\hat{\alpha}' \hat{\alpha}$ v r_j v
SS_Y	$Y'Y$ v N v	$Y'Y$ v N v
SS_e	$Y'Y - Y.'D Y.$ v N v v c c v	$Y'Y - \hat{\alpha}'\hat{\alpha}$ v N v v r v

Note.- Y is the original subject by variable data matrix.

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